Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 2229–2233
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

A counterexample to a conjecture due to Douglas, Reinbacher and Yau

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Received 23 October 2006; received in revised form 26 June 2007; accepted 3 July 2007
Available online 15 August 2007

Abstract

In "Branes, Bundles and Attractors: Bogomolov and Beyond", by Douglas, Reinbacher and Yau, the authors state the following conjecture: Consider a simply connected surface X with ample or trivial canonical line bundle. Then, the Chern classes of any stable vector bundle with rank $r \geq 2$ satisfy $2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(X) \geq 0$. The goal of this short note is to provide two sources of counterexamples to this strong version of the Bogomolov inequality.

MSC: 14F05

Keywords: Chern classes; Stable; Vector bundles

1. Introduction

In [2] Douglas, Reinbacher and Yau state several conjectures arising from the attractor mechanism in type II string theory concerning possible Chern classes of stable vector bundles on algebraic varieties. In particular, the paper contains the following conjecture which is a slight strengthening of the Bogomolov inequality.

Conjecture 1.1. *Consider a simply connected surface X with ample or trivial canonical bundle and let H be an ample line bundle on X. Then, the Chern classes of any μ_H-stable vector bundle of rank $r \geq 2$ satisfy*

$$2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(X) \geq 0.$$

On the basis of physical evidence, this conjecture was first stated for Kähler manifolds of dimension n in a preliminary version of Douglas, Reinbacher and Yau's paper. In [5] Jardim provides examples that show that the conjecture does not hold for stable vector bundles on Calabi–Yau threefolds. In a revised version of the paper of Douglas, Reinbacher and Yau, the original conjecture was replaced by the above statement concerning the Chern classes of stable vector bundles on simply connected surfaces with ample or trivial canonical bundle. The goal of this

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.07.001

paper is to prove that the reformulated version is also false. We will provide two kinds of examples. The first one (see Proposition 2.1) concerns rank $r \geq 2$ vector bundles on a generic $K3$ surface X (i.e. on a generic algebraic surface X with $q(X) = 0$ and trivial canonical line bundle). The second one (see Proposition 3.2) is devoted to rank $r \geq 3$ vector bundles on a surface X in $\mathbb{P}^3$ of degree $d \geq 7$ (and hence its canonical line bundle is ample).

Terminology: Let H be an ample line bundle on a smooth projective algebraic surface X. For a torsion free sheaf F on X we set

$$\mu(F) = \mu_H(F) := \frac{c_1(F)H}{rk(F)}.$$

The sheaf F is said to be μ_H-*semistable* if

$$\mu_H(E) \leq \mu_H(F)$$

for all non-zero subsheaves $E \subset F$ with $rk(E) < rk(F)$; if strict inequality holds then F is μ_H-*stable*. Notice that for rank r vector bundles F on X with $(c_1(F)H, r) = 1$, the concepts of μ_H-stability and μ_H-semistability coincide.

Recall that for any rank r vector bundle F on a cyclic variety X with $\mathrm{Pic}(X)$ generated by h, there is a uniquely determined integer k_F such that if $c_1(F(k_F h)) = c_1 h$, then $-r + 1 \leq c_1 \leq 0$. We set $F_{\mathrm{norm}} = F(k_F h)$.

2. First example

The goal of this section is to see that Conjecture 1.1 fails for μ_H-stable rank 2 vector bundles on generic $K3$ surfaces with trivial canonical line bundle. To this end, let X be a complex algebraic $K3$ surface, that is X is a complete regular surface with trivial canonical line bundle and irregularity $q(X) = 0$. According to [7], we will say that a vector bundle E on X is *exceptional* if

$$\dim \mathrm{Hom}(E, E) = 1 \quad \text{and} \quad \mathrm{Ext}^1(E, E) = 0,$$

i.e., E is simple and rigid. Any coherent sheaf F on X has associated a *Mukai vector*

$$v(F) = \left(r, c_1, \frac{c_1^2}{2} + r - c_2\right)$$

where $r = \mathrm{rank}(F)$ and c_1, c_2 denote the first and second Chern classes of F. A Mukai vector is called exceptional if according to the inner product defined in the Mukai lattice (see [8]) the following equality holds:

$$v(F)^2 = c_1^2 - 2r\left(r - c_2 + \frac{c_1^2}{2}\right) = -2.$$

When X is a $K3$ surface, $c_2(X) = 24$. Indeed we have

$$2 = \chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + c_2(X)) = \frac{c_2(X)}{12}.$$

So, in that case Conjecture 1.1 is equivalent to saying that the Chern classes of any μ_H-stable vector bundle of rank $r \geq 2$ satisfy

$$2rc_2 - (r-1)c_1^2 - 2r^2 \geq 0.$$

Let us see that there exist infinitely many μ_H-stable vectors on a generic $K3$ surface whose Chern classes do not satisfy the inequality above. First of all notice that if X is a generic $K3$ surface then $\mathrm{Pic}(X) \cong \mathbb{Z}$.

Proposition 2.1. *Let X be a generic $K3$ surface and let H be an arbitrary ample line bundle on X. For any Mukai vector $v = (r, c_1, \frac{c_1^2}{2} + r - c_2)$ such that $(r, c_1 H) = 1$ and*

$$2rc_2 - (r-1)c_1^2 = 2r^2 - 2$$

there exists a μ_H-stable rank r vector bundle E on X with Mukai vector $v(E) = v$.

Proof. By [7], Theorem 2.1, for any Mukai exceptional vector $v = (r, c_1, \frac{c_1^2}{2} + r - c_2)$, there exists a μ_H-semistable vector bundle E on X with $v(E) = v$. By assumption, $(r, c_1H) = 1$, so the notions of μ_H-semistability and μ_H-stability coincide. Following the definition of Mukai exceptional vector, we get that for any of these vectors $v = (r, c_1, \frac{c_1^2}{2} + r - c_2)$ such that $(r, c_1H) = 1$ and

$$2rc_2 - (r-1)c_1^2 = 2r^2 - 2$$

there exists a μ_H-stable rank r vector bundle E on X with Mukai vector $v(E) = v$. □

Therefore we have proved the existence of infinitely many μ_H-stable vector bundles E on a generic $K3$ surface with Chern classes contradicting the inequality predicted in Conjecture 1.1.

3. Second example

The goal of this section is to see that Conjecture 1.1 fails for μ_H-stable rank $r \geq 3$ vector bundles on degree $d \geq 7$ surfaces in $\mathbb{P}^3$. As a main tool we will use the theory of monads introduced by Horrocks in [4] and developed by the authors in [1]. In order to do that, let X be a surface of degree $d \geq 7$ in $\mathbb{P}^3$ and denote by h the restriction to X of the hyperplane section H of $\mathbb{P}^3$. Recall that the Picard group of X is generated by h, $K_X = (d-4)h$ is ample, $h^2 = d$ and $K_X^2 = (d-4)^2 d$. In addition,

$$P_g(X) = \frac{(d-1)(d-2)(d-3)}{6} \quad \text{and} \quad P_g(X) + 1 = \frac{1}{12}(K_X^2 + c_2(X)).$$

Lemma 3.1. *For any integer $c \geq 2$, there exists a monad on X of the following type:*

$$M_\bullet: \quad 0 \longrightarrow \mathcal{O}_X(-h)^{c-1} \xrightarrow{\alpha} \mathcal{O}_X^{2c+1} \xrightarrow{\beta} \mathcal{O}_X(h)^c \longrightarrow 0$$

whose cohomology sheaf $E = \mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ is a rank 2 vector bundle on X.

Proof. Set $\mathbb{P}^3 = \mathrm{Proj}(k[x_0, x_1, x_2, x_3])$. Without loss of generality we may assume that X is the surface in $\mathbb{P}^3$ defined by $f(x_0, \ldots, x_3) = x_0^d + x_1^d + x_2^d + x_3^d$. Consider the $(c+1) \times c$, $c \times c$, and $(c+1) \times (c+1)$ matrices given by

$$A_1 = \begin{pmatrix} x_0 & x_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & x_0 & x_1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & x_0 & x_1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} x_2 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & x_2 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} x_2 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & x_2 \end{pmatrix}.$$

Define the complex

$$0 \longrightarrow \mathcal{O}_X(-h)^{c+1} \xrightarrow{\gamma} \mathcal{O}_X^{2c+1} \xrightarrow{\beta} \mathcal{O}_X(h)^c \longrightarrow 0 \tag{3.1}$$

where β is the map given by the matrix $B = (A_1 A_2)$ and γ is the map given by

$$A = \begin{pmatrix} A_3 \\ -A_1 \end{pmatrix}.$$

It is not difficult to see that γ degenerates in codimension 1. Now consider a sufficiently general injection $\phi : \mathcal{O}_X(-h)^{c-1} \longrightarrow \mathcal{O}_X(-h)^{c+1}$ and its composition with the map γ defined in (3.1). If ϕ is general enough, $\gamma\phi$ degenerates in codimension 3. Hence, by [6], Proposition 4, we get a monad

$$M_\bullet: \quad 0 \longrightarrow \mathcal{O}_X(-h)^{c-1} \xrightarrow{\gamma\phi} \mathcal{O}_X^{2c+1} \xrightarrow{\beta} \mathcal{O}_X(h)^c \longrightarrow 0$$

whose cohomology sheaf $E = \mathrm{Ker}(\beta)/\mathrm{Im}(\gamma\phi)$ is a rank 2 vector bundle. □

For any integer $c \geq 2$, let

$$M_\bullet: \quad 0 \longrightarrow \mathcal{O}_X(-h)^{c-1} \xrightarrow{\alpha} \mathcal{O}_X^{2c+1} \xrightarrow{\beta} \mathcal{O}_X(h)^c \longrightarrow 0$$

be a monad given by Lemma 3.1. Denote by $T = \mathrm{Ker}(\beta)$ and by K its dual. Since T is a rank $c+1$ vector bundle on X, K is also a rank $c+1$ vector bundle on X and has Chern classes $c_1(K) = ch$ and $c_2(K) = \frac{c(c+1)d}{2}$. For any $c \geq 2$ and $d \geq 7$,

$$2c^2 + c < (c^2 + 2c + 1)\frac{d^2 - 4d + 6}{12}$$

and hence

$$2\mathrm{rk}(K)c_2(K) - (\mathrm{rk}(K) - 1)c_1(K)^2 < \frac{\mathrm{rk}(K)^2}{12}c_2(X).$$

Let us see that K is stable. By applying Hoppe's criterion ([3], Lemma 2.6), it is enough to see that

$$H^0\left(\left(\bigwedge^q K\right)_{\mathrm{norm}}\right) = 0, \quad 1 \leq q \leq rk(K) - 1 = c.$$

First of all let us consider the case $q = 1$. Since $c_1(K) = ch$, $(K)_{\mathrm{norm}} = K(th)$ for some $t \leq -1$. Hence it is enough to see that $H^0(K(-h)) = 0$ which easily follows from the cohomological exact sequence associated with

$$0 \longrightarrow \mathcal{O}_X(-h)^c \longrightarrow \mathcal{O}_X^{2c+1} \longrightarrow K \longrightarrow 0. \tag{3.2}$$

Let us now assume $2 \leq q \leq c$. Notice that for any $0 \leq t \leq c-2$,

$$\mu_h\left(\left(\bigwedge^{t+2} K\right)((-t-1)h)\right) = (2+t)\mu_h(K) - (t+1)h^2 = \frac{(c-t-1)}{c+1}d > 0.$$

Hence $(\bigwedge^{t+2} K)_{\mathrm{norm}} = (\bigwedge^{t+2} K)(jh)$ for some $j \leq -t-2$ and thus it is enough to see that

$$H^0\left(\left(\bigwedge^{t+2} K\right)((-t-2)h)\right) = 0. \tag{3.3}$$

We will prove (3.3) by induction on t. Let us assume $t = 0$. The display of the monad $M_\bullet$ gives us the following two short exact sequences:

$$0 \longrightarrow \mathcal{O}_X(-h)^c \longrightarrow \mathcal{O}_X^{2c+1} \longrightarrow K \longrightarrow 0, \tag{3.4}$$

$$0 \longrightarrow E^* \longrightarrow K \longrightarrow \mathcal{O}_X(h)^{c-1} \longrightarrow 0 \tag{3.5}$$

where by Lemma 3.1, E is a rank 2 vector bundle on X. The second exterior power of the exact sequence (3.5) twisted by $\mathcal{O}_X(-2h)$ gives us the following long exact sequence:

$$0 \longrightarrow \bigwedge^2(E^*)(-2h) \longrightarrow \bigwedge^2(K)(-2h) \longrightarrow K(-h)^{c-1} \longrightarrow S^2(\mathcal{O}_X(h)^{c-1})(-2h) \longrightarrow 0. \tag{3.6}$$

Since E is a rank 2 vector bundle,

$$H^0\left(\bigwedge^2(E^*)(-2h)\right) = H^0(\mathcal{O}_X(c_1(E^*) - 2h)) = H^0(\mathcal{O}_X(-h)) = 0$$

and for the case $q = 1$ we have $H^0(K(-h)) = 0$. Thus, using the exact sequence (3.6) we deduce that $H^0((\bigwedge^2 K)(-2h)) = 0$ which finishes the case $t = 0$. For $t > 0$, twisting by $\mathcal{O}_X((-2-t)h)$ the $(t+2)$-exterior power of the exact sequence (3.5), we get the long exact sequence

$$0 \longrightarrow \bigwedge^{2+t}(K)((-2-t)h) \longrightarrow \bigwedge^{1+t}(K)((-1-t)h)^{c-1} \longrightarrow \cdots.$$

By the inductive hypothesis $H^0((\bigwedge^{t+1} K)((-t-1)h)) = 0$. Therefore,

$$H^0\left(\left(\bigwedge^{t+2} K\right)((-t-2)h)\right) = 0$$

which concludes the proof of the stability of K.

Putting this together we get the following result:

Proposition 3.2. *Let X be a smooth surface of degree $d \geq 7$ in $\mathbb{P}^3$. Then, there exists a rank $r \geq 3$ vector bundle F on X with Chern classes $c_1(F) = c_1$ and $c_2(F) = c_2$ verifying*

$$2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(X) < 0.$$

Proof. Set F to be equal to the vector bundle K from above. □

Notice that since any smooth surface $X \subset \mathbb{P}^3$ of degree $d \geq 7$ is an algebraic surface with ample canonical line bundle, Proposition 3.2 provides us with an infinite family of examples contradicting Conjecture 1.1.

4. Final remark

Notice that the vector bundles E given in Proposition 2.1 are points of a trivial moduli space, that is of a zero-dimensional moduli space. On the other hand, vector bundles given in Proposition 3.2 are points of a nontrivial moduli space. Indeed, following the above notation let us prove that $\mathrm{Ext}^1(K, K) \neq 0$. Let us assume that $\mathrm{Ext}^1(K, K) = 0$. Twisting by K the short exact sequence

$$0 \longrightarrow K^* \longrightarrow \mathcal{O}_X^{2c+1} \longrightarrow \mathcal{O}_X(h)^c \longrightarrow 0$$

and taking cohomology we get

$$0 \longrightarrow H^0(K^* \otimes K) \longrightarrow H^0(\mathcal{O}_X^{2c+1} \otimes K) \longrightarrow H^0(\mathcal{O}_X(h)^c \otimes K) \longrightarrow H^1(K^* \otimes K) \to \cdots.$$

We know that K is a stable vector bundle, and hence it is simple, i.e. $h^0(K \otimes K^*) = 1$, and by assumption $0 = \mathrm{Ext}^1(K, K) = H^1(K \otimes K^*)$. Thus

$$(2c+1)h^0(K) = 1 + ch^0(K(h)). \tag{4.1}$$

On the other hand, using the exact sequence (3.2) we deduce that $h^0(K) = 2c + 1$ and $h^0(K(h)) = 7c + 4$ which contradicts (4.1). Therefore, $\mathrm{Ext}^1(K, K) \neq 0$ and indeed the corresponding moduli space is nontrivial.

We want to point out that with Proposition 3.2 we not only provide counterexamples to the quoted conjecture of Douglas, Reinbacher and Yau but also provide counterexamples to a reformulated version of the Conjecture 1.1 stated recently by the same authors in the fourth version of [2].

Acknowledgments

The authors thank C. D'Andrea for his valuable comments.

The first author was partially supported by MTM2004-00666 and the second author was partially supported by MTM2004-00666.

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